Quantum Physics B

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Exam, Tuesday January 27, 2004
4 problems (total of 50 points).
The solution of every problem on a separate piece of paper with name and student number.
Use the attached formula list where necessary.

Problem 1 (15 pnts in total)

The electron in a hydrogen atom occupies the combined spin and position state

$$\Psi = R_{21}\left(\sqrt{3}Y_1^{-1}\chi_+ + \sqrt{2}Y_1^0\chi_-\right)/\sqrt{5} \tag{1}$$

1 pnts a. If you measured the orbital angular momentum squared (L^2), what values might you get, and what is the probability of each?

1 pnts b. Same for the z-component of orbital angular momentum (L_z).

1 pnts c. Same for the z-component of spin angular momentum (S_z).

1 pnts d. Same for the z-component of total angular momentum, $J_z = L_z + S_z$.

3 pnts e. Calculate for this wave function the expectation value $< \Psi|\vec{S}\cdot\vec{n}|\Psi >$ where $\vec{n} = \hat{x}\cos\alpha + \hat{z}\sin\alpha$.

2 pnts f. If you measured J^2, what values might you get and what is the probability of each? (you may use the table of Clebsch-Gordan coefficients).

3 pnts g. Calculate $\Phi = J_-\Psi$ where $J_- = L_- + S_-$.

3 pnts h. In an experiment one measures r, the distance to the origin, as well as m_s, the z-projection of the electron spin. Give the probability density to find the electron with $m_s = -1/2$ at a distance r.

Problem 2 (10 pnts in total)

An electron in the $n = 3$, $l = 0$, $m = 0$ state of hydrogen decays by a sequence of (electric dipole) transitions to the ground state.

3 pnts a. What decay modes are open to it? Specify them in the following way:

$$|300> \to |nlm> \to |n'l'm'> \to \cdots \to |100> \ .$$

2 pnts b. If you had a bottle full of atoms in this state, what fraction of them would decay via each route?

5 pnts c. What is the lifetime of this state.

Problem 3 (15 pnts in total)

1 pnts a. Give the time-independent Schrödinger equation for the problem of an infinite square well in two dimensions,

$$V(x,y) = \begin{cases} 0 & \text{if } 0 < x < a \text{ and } 0 < y < a \\ \infty & \text{otherwise} \end{cases} .$$

Specify also the boundary conditions.

3 pnts b. Use separation of variables to solve for the energies and wave functions.

2 pnts c. What is the degeneracy of the ground state and the first excited state?

4 pnts d. We add the perturbation H' to the Hamiltonian,

$$H' = \begin{cases} 2V_0, & \text{if } 0 < x < a/4 \text{ and } 0 < y < a/4 \\ V_0, & \text{if } a/4 < x < a/2 \text{ and } a/4 < y < a/2 \\ 0, & \text{other wise} \end{cases} .$$

Give the first order perturbation correction to the energy of the ground state.

5 pnts e. Give the first order perturbation correction to the energies of the first excited states which were degenerate without perturbation.

Problem 4 (10 pnts in total)

An electron is at rest at the origin in the presence of a magnetic field whose magnitude (B_0) is constant but whose direction rotates around in the (x,y) plane at constant angular velocity α

$$\vec{B}(t) = B_0 \left[\cos(\alpha t)\hat{x} + \sin(\alpha t)\hat{y}\right] . \tag{2}$$

The hamiltonian for the particle is now given by $H = (e/m)\vec{B} \cdot \vec{S}$, where $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ are the spin matrices. A possible solution is given by the spinor

$$\chi(t) = \begin{pmatrix} [\cos(\lambda t/2) + i(\alpha/\lambda)\sin(\lambda t/2)]e^{-i\alpha t/2} \\ i(\omega/\lambda)\sin(\lambda t/2)e^{i\alpha t/2} \end{pmatrix} \tag{3}$$

where $\omega = -eB_0/m$ and $\lambda = \sqrt{\alpha^2 + \omega^2}$.

2 pnts a. Write the Hamiltonian explicitly as a 2×2 matrix.

3 pnts b. Show that $\chi(t)$ is indeed a solution of the time-dependent Schrödinger equation for this problem.

1 pnts c. Verify that $\chi(t)$ is normalized.

2 pnts d. Calculate $< \sigma_z >$ to verify that $\chi(t = 0)$ corresponds to a spin-up electron.

2 pnts e. Calculate the expectation value of the spin in the y-direction as a function of time.

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The following formula's may be helpful in solving the problems.

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Sigma (spin) matrices.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

$$\sigma_{x,y,z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \tag{3}$$

Harmonic oscillator wave functions.

Solutions for a harmonic oscillator potential $V(x) = \frac{\omega^2 m}{2}x^2$

$$u_n = (\frac{m\omega}{\pi\hbar})^{1/4}(2^n n!)^{-1/2}H_n(y)e^{-y^2/2} \tag{4}$$

with $y = \sqrt{m\omega/\hbar}\, x$, where the Hermiet polynomials for $n \leq 4$ are given as

$$H_0(y) = 1 \tag{5}$$
$$H_1(y) = 2y \tag{6}$$
$$H_2(y) = 4y^2 - 2 \tag{7}$$
$$H_3(y) = 8y^2 - 12y \tag{8}$$
$$H_4(y) = 16y^4 - 48y^2 + 12 \tag{9}$$

Matrix elements:

$$< n|x^2|n > = < n|p^2|n > /(m\omega)^2 = (2n+1)\,\frac{\hbar}{2m\omega} \tag{10}$$

$$< n|x^2|n-2 > = -< n|p^2|n-2 > /(m\omega)^2 = \sqrt{n(n-1)}\,\frac{\hbar}{2m\omega} \tag{11}$$

$$< n|x^3|n-1 > = 3\,n^{3/2}\left(\frac{\hbar}{2m\omega}\right)^{3/2} \tag{12}$$

$$< n|x^3|n-3 > = \sqrt{n(n-1)(n-2)}\left(\frac{\hbar}{2m\omega}\right)^{3/2} \tag{13}$$

$$< n|x^4|n > = [2(n+1)(n+2) + (2n-1)(2n+1)]\left(\frac{\hbar}{2m\omega}\right)^2 \tag{14}$$

$$< n|x^4|n-2 > = 2(2n-1)\sqrt{n(n-1)}\left(\frac{\hbar}{2m\omega}\right)^2 \tag{15}$$

$$< n|x^4|n-4 > = \sqrt{n(n-1)(n-2)(n-3)}\left(\frac{\hbar}{2m\omega}\right)^2 \tag{16}$$

Hydrogen wave functions.

$R_{nl}(r)$ are hydrogen-like wave functions with $E_n = -\alpha^2 m_e c^2/2n^2 = -13.6\ eV/n^2$, $a_0 = \hbar/m_e c\alpha$ and $\alpha = e^2/\hbar c = 1/137$.

$$R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\ , \tag{17}$$

$$R_{20}(r) = 2\left(\frac{Z}{2a_0}\right)^{3/2}\left(1-\frac{Zr}{2a_0}\right) e^{-Zr/2a_0}\ , \tag{18}$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{a_0}\, e^{-Zr/2a_0}\ , \tag{19}$$

$$R_{32}(r) = \frac{8}{81\sqrt{15}}\left(\frac{Z}{2a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)^2 e^{-Zr/3a_0}\ . \tag{20}$$

Spherical harmonics Y_l^m.

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \;; Y_1^1 = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi}\, \sin\theta \;; Y_1^0 = \sqrt{\frac{3}{4\pi}}\, \cos\theta \;, \tag{21}$$

$$Y_2^2 = \sqrt{\frac{15}{32\pi}}\, e^{2i\phi}\, \sin^2\theta \;; Y_2^1 = -\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\, \sin\theta\, \cos\theta \;; Y_2^0 = \sqrt{\frac{5}{16\pi}}\, (3\cos^2\theta - 1) \;, \tag{22}$$

with $Y_l^{-m} = (-1)^m [Y_l^m]^*$, and the normalization condition:

$$\int d\Omega\, [Y_l^m(\Omega)] * Y_{l'}^{m'}(\Omega) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, [Y_l^m(\Omega)]^* Y_{l'}^{m'}(\Omega) = \delta_{l,l'}\, \delta_{m,m'} \;. \tag{23}$$

$$L_+ = L_x + iL_y \quad \text{and} \quad L_+\, Y_l^m = \hbar\sqrt{l(l+1) - m(m+1)}\; Y_l^{m+1} \;, \tag{24}$$

$$L_- = L_x - iL_y \quad \text{and} \quad L_-\, Y_l^m = \hbar\sqrt{l(l+1) - m(m-1)}\; Y_l^{m-1} \;. \tag{25}$$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = L_+ L_- + L_z^2 - \hbar L_z = L_- L_+ + L_z^2 + \hbar L_z$$

In addition:

$$|l, j, m_j> \;=\; \sqrt{\frac{l-m}{2l+1}} |Y_l^{m+1}\chi_- > + \sqrt{\frac{l+m+1}{2l+1}} |Y_l^m \chi_+ > \quad \text{for } j = l + 1/2 \tag{26}$$

$$|l, j, m_j> \;=\; \sqrt{\frac{l+m+1}{2l+1}} |Y_l^{m+1}\chi_- > - \sqrt{\frac{l-m}{2l+1}} |Y_l^m \chi_+ > \quad \text{for } j = l - 1/2 \tag{27}$$

with $m = m_j - 1/2$.

Table 4.7: Clebsch-Gordan coefficients. (A square root sign is understood for every entry; the minus sign, if present, goes *outside* the radical.)

1/2 × 1/2

	1 +1
+1/2 +1/2	1

	1 0	0 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	1 −1
−1/2 −1/2	1

1 × 1/2

	3/2 +3/2
+1 +1/2	1

	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

	3/2 −3/2
−1 −1/2	1

2 × 1

	3 +3
+2 +1	1

	3 +2	2 +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

	3 +1	2 +1	1 +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

	3 0	2 0	1 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

	3 −1	2 −1	1 −1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

	3 −2	2 −2
−1 −1	2/3	1/3
2 0	1/3	−2/3

	3 −3
−2 −1	1

1 × 1

	2 +2
+1 +1	1

	2 +1	1 +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

	2 0	1 0	0 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

	2 −1	1 −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

	2 −2
−1 −1	1

3/2 × 1

	5/2 +5/2
+3/2 +1	1

	5/2 +3/2	3/2 +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2 −3/2	3/2 −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2 −5/2
−3/2 −1	1

2 × 1/2

	5/2 +5/2
+2 1/2	1

	5/2 3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2 −5/2
−2 −1/2	1

3/2 × 1/2

	2 +2
+3/2 +1/2	1

	2 +1	1 +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

	2 0	1 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	2 −1	1 −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

	2 −2
−3/2 −1/2	1

$$\int_{-a}^{a} e^{i\alpha x}\,dx = \frac{2}{\alpha}\sin(\alpha a)\,, \tag{28}$$

$$\int_{-a}^{a} \cos\alpha x\, e^{ikx}\,dx = \left[\frac{\sin(\alpha+k)a}{\alpha+k} + \frac{\sin(\alpha-k)a}{\alpha-k}\right]\,, \tag{29}$$

$$\int_{-a}^{a} \sin\alpha x\, e^{ikx}\,dx = i\left[\frac{\sin(\alpha+k)a}{\alpha+k} - \frac{\sin(\alpha-k)a}{\alpha-k}\right]\,, \tag{30}$$

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx = 2\pi\,\delta(k-k')\,, \tag{31}$$

$$\int_{-\infty}^{\infty} f(p')\,\delta(p-p')\,dp' = f(p) \quad (\text{mits } f(p) \text{ differentieerbaar in } p)\,, \tag{32}$$

$$\int_{-\infty}^{\infty} e^{-a(x+b+ic)^2}dx = \sqrt{\pi/a}\,, \tag{33}$$

$$\int_{-\infty}^{\infty} x^2 e^{-a(x+b)^2}dx = (b^2+1/2a)\sqrt{\pi/a}\,, \tag{34}$$

$$\int_{-\infty}^{\infty} e^{-a(x+b)^2}e^{ikx}dx = \sqrt{\pi/a}\,e^{-ikb-k^2/4a}\,, \tag{35}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\cos(bx)\,dx = \sqrt{\pi/a}\,e^{-b^2/4a}\,, \tag{36}$$

$$\int_{0}^{\infty} x^{2n}e^{-ax^2}dx = \tfrac{1}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2a)^n}\sqrt{\pi/a} \text{ voor } n \geq 0\,, \tag{37}$$

$$= \tfrac{1}{2}\sqrt{\frac{\pi}{a}} \quad \text{voor} \quad n=0\,, \tag{38}$$

$$\int_{0}^{\infty} x^{2n+1}e^{-ax^2}dx = \frac{n!}{2\,a^{n+1}} \quad \text{met} \quad a>0\,, \tag{39}$$

$$\int_{0}^{\infty} x^{n}e^{-ax}dx = \frac{n!}{a^{n+1}} \quad \text{met} \quad a>0\,, \tag{40}$$

$$\int_{0}^{\infty} xe^{-ax}\sin(bx)\,dx = \frac{2ab}{(a^2+b^2)^2} \quad \text{met} \quad a>0\,, \tag{41}$$

$$\int_{0}^{\infty} xe^{-ax}\cos(bx)\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} \quad \text{met} \quad a>0\,, \tag{42}$$

$$\int_{0}^{\infty} \frac{\sin^2(px)}{x^2}dx = \tfrac{1}{2}\pi p\,, \tag{43}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,e^{ikx}dx = \frac{\pi}{a}e^{-a|k|}\,, \text{ ook geldig voor k=0}\,, \tag{44}$$

$$\int_{0}^{a} x^2\,\sin^2 n\pi x/a\,dx = \frac{a^3}{4}\left[\frac{2}{3} - \frac{1}{(n\pi)^2}\right]\,, \tag{45}$$

$$\int_{0}^{a} x^2\,\cos^2(n-\frac{1}{2})\pi x/a\,dx = \frac{a^3}{4}\left[\frac{2}{3} - \frac{1}{((n-\frac{1}{2})\pi)^2}\right]\,, \tag{46}$$

$$\int_{0}^{\pi} \sin^m\theta\,d\theta = \sqrt{\pi}\,\Gamma(\frac{m+1}{2})/\Gamma(\frac{m+2}{2})\,, \tag{47}$$

$$\int_{0}^{\infty} \frac{x^a}{(x^b+q^b)^c}dx = \frac{q^{a+1-bc}}{b}\frac{\Gamma(\frac{a+1}{b})\Gamma(c-\frac{a+1}{b})}{\Gamma(c)}\,, \tag{48}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{2\,a^3}\,, \tag{49}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^n}dx = \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\,\frac{\pi}{a^{2n-1}} \text{ voor } n\geq 2, \tag{50}$$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)! \quad ; \quad \Gamma(1) = 0! = 1\,, \tag{51}$$

$$\Gamma(n+\tfrac{1}{2}) = 2^{-n}[1\cdot 3\cdot 5\cdots(2n-1)]\sqrt{\pi} \quad ; \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad ; \quad \Gamma(\tfrac{3}{2}) = \sqrt{\pi}/2\,, \tag{52}$$

$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x)\,. \tag{53}$$